The vertex formulation of the Bazhanov – Baxter model

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Abstract

In this paper we formulate an integrable model on the simple cubic lattice. The N – valued spin variables of the model belong to edges of the lattice. The Boltzmann weights of the model obey the vertex type Tetrahedron Equation. In the thermodynamic limit our model is equivalent to the Bazhanov – Baxter Model. In the case when $N = 2$ we reproduce the Korepanov's and Hietarinta's solutions of the Tetrahedron equation as some special cases.

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1. Introduction

Recently two new solutions of the vertex type Tetrahedron equation [1–3] for the number of spin states $N = 2$ [4, 5] were obtained. In our previous paper we have tried to generalize these solutions for $N > 2$ and for general spectral parameters. We have succeeded there in a generalization of the solution from Ref. [5] for arbitrary N.

For the case $N = 2$ the solution proposed by Hietarinta appears to be a some special case of the planar limit of the Bazhanov – Baxter solution [6]. Recall that in the Bazhanov – Baxter model (BBM) [7] N – valued spin variables belong to the vertices of the elementary cubes of the lattice and the Boltzmann weights in the Tetrahedron Equation (TE) are parameterized by the angles of a tetrahedron [8].

As is known the Bazhanov-Baxter model can not be directly reformulated as a vertex type model using an obvious duality between vertex and interaction-round-cube formulations. For example, for the case $N = 2$ (Zamolodchikov model) [1] such a duality requires an invariance of the weight functions with respect to a recolouring of any face of the elementary cube. It is known [9] that the Boltzmann weights of Zamolodchikov model in general do not possess this symmetry (despite the fact that the absolute values of the Boltzmann weights do).

Nevertheless in the particular limit when all four vertices of the tetrahedron belong to the same plane (the planar limit), it is possible to rewrite the Boltzmann weights using 2-state edge variables only and as a result to obtain the vertex solution of the TE from Ref. [5]. Note that for $N > 2$ the solution of the TE from Ref. [6] does not coincide with the planar limit of the BBM and seems to be new.

Attempts to remove the planar limit restriction for this solution have been failed. Instead we have obtained a complete (depending on three arbitrary angles) vertex solution of the TE for general number of spin variables N. This solution at $N = 2$ reproduces the solutions of Korepanov and Hietarinta in the static and planar limits correspondingly. This new model in the thermodynamic limit coincides with the BBM. However, due to the vertex form, this formulation may be useful for a more careful investigation of the model. Namely, one can try to formulate the Bethe – ansatz, construct a functional equation for the transfer matrices analogically to the two – dimensional case. Also one can try to construct a three dimensional generalization of the L

operators and etc.

The paper is organized as follows. In the second section we recall usual notations for the functions on Z_N which will be used for the constructing of Boltzmann weights. In the third section we give an explicit form of the vertex weight function and show an equivalence of our vertex model with the BBM in the thermodynamic limit. Symmetry properties of the vertex weight we list in the fourth section. Also we give some exotic forms of the gauges and write out the inversion relation for the weight functions. The case $N = 2$ is considered in some special gauge in the fifth section, where we show the equivalence of our vertex weight in the static limit with the solution of the TE proposed by Korepanov. The sixth section is devoted to the sketch of the proof of the TE for the vertex weight. At last, in Appendix we collect the most useful formulae for ω – hypergeometric seria with ω being a N-th root of unity.

2. Notations and definitions.

In this section we give a list of all necessary definitions and notations.

Denote

$$\omega^{1/2} = \exp(\pi i/N), \quad N \in Z. \tag{2.1}$$

Let x, y, z are three homogeneous complex variables constrained by Fermat equation

$$x^N + y^N = z^N. \tag{2.2}$$

Hereafter we use a notation $p = (x, y, z)$ anywhere unless it will lead to misunderstanding.

Now we define a function $w(p|a)$ by recurrence relation:

$$\frac{w(p|a)}{w(p|0)} = \prod_{s=1}^{a} \frac{y}{z - x\omega^s}, \tag{2.3}$$

where a is an element of Z_N.

Following Ref. [10] we will choose a normalization factor $w(p|0)$ as follows. First let us set $z = 1$ and consider the case $|x| < 1$. Then we can choose y as

$$y = (1 - x^N)^{1/N}. \tag{2.4}$$

For such x, y and z we put

$$w(p|0) = y^{(1-N)/2} \prod_{j=1}^{N-1} (1-\omega^{-j}x)^{j/N}. \tag{2.5}$$

With such a normalization the function $w(p|a)$ satisfies

$$\prod_{a=0}^{N-1} w(p|a) = 1. \tag{2.6}$$

Further we can analytically continue formulas (2.4-2.6) over x into the whole complex plane with cuts from the points $x = \omega^n$, $n = 0, \ldots, N-1$ to infinity. For such x and y we will say that the point $p = (x, y, 1)$ belongs to the main branch Γ_0 of some covering curve Γ on which the function $w(p|a)$ is well defined. If we go under the cut around the point $x = \omega^n$ in the anti-clockwise direction, then $w(p|0)$ is multiplied on the phase factor $(-1)^{N-1}\omega^n$. Restoring z dependence it is easy to check that

$$w(\omega^n x, y, z|m) = w(x, y, z|m+n), \quad m, n \in Z_N,\ p = (x, y, z) \in \Gamma_0. \tag{2.7}$$

Now let us consider a region on Γ_0 such that

$$-2\pi/N < \mathrm{Arg}(x/z) < 0 \tag{2.8}$$

and denote it as $\overline{\Gamma}_0$. It is easily to show that for the points $p \in \overline{\Gamma}_0$ we have

$$-\pi/N < \mathrm{Arg}(y/z) < \pi/N. \tag{2.9}$$

Then for the given point $p = (x, y, z) \in \overline{\Gamma}_0$ define a new point $Op \in \overline{\Gamma}_0$ as

$$Op = (z, \omega^{1/2} y, \omega x). \tag{2.10}$$

Using these notations we have the following property of the w function:

$$w(p|a)w(Op|-a)\Phi(a)\exp(\frac{i\pi(N^2-1)}{6N}) = 1, \quad a \in Z_N,\ p \in \overline{\Gamma}_0, \tag{2.11}$$

where

$$\Phi(a) = \omega^{a(a-N)/2}. \tag{2.12}$$

In Appendix we will give a set of the most useful formulae and identities for the w function.

3. The vertex weights.

For a given spherical triangle with the angles $\theta_1, \theta_2, \theta_3$ and corresponding linear angles (i.e. sides of the spherical triangle) a_1, a_2, a_3 define four points $p_i \in \overline{\Gamma}_0$:

$$
\begin{gathered}
x(p_1) = \omega^{-1/2} \exp(i\frac{a_3}{N}) \sqrt[N]{\frac{\sin\beta_1}{\sin\beta_2}},\ y(p_1) = \exp(i\frac{\beta_1}{N}) \sqrt[N]{\frac{\sin a_3}{\sin\beta_2}};\\
x(p_2) = \omega^{-1/2} \exp(i\frac{a_3}{N}) \sqrt[N]{\frac{\sin\beta_2}{\sin\beta_1}},\ y(p_2) = \exp(i\frac{\beta_2}{N}) \sqrt[N]{\frac{\sin a_3}{\sin\beta_1}};\\
x(p_3) = \omega^{-1} \exp(i\frac{a_3}{N}) \sqrt[N]{\frac{\sin\beta_3}{\sin\beta_0}},\ y(p_3) = \exp(-i\frac{\beta_3}{N}) \sqrt[N]{\frac{\sin a_3}{\sin\beta_0}};\\
x(p_4) = \omega^{-1} \exp(i\frac{a_3}{N}) \sqrt[N]{\frac{\sin\beta_0}{\sin\beta_3}},\ y(p_4) = \exp(-i\frac{\beta_0}{N}) \sqrt[N]{\frac{\sin a_3}{\sin\beta_3}};\\
z(p_i) = 1,\ i = 1,2,3,4;
\end{gathered}
\tag{3.1}
$$

where β_i being usual linear excesses

$$\beta_0 = \pi - \frac{a_1 + a_2 + a_3}{2},\ \beta_i = \pi - \beta_0 - a_i. \tag{3.2}$$

Further we will consider (3.1) as a definition for the points $p_i \equiv p_i(a_1, a_2, a_3)$.

Let $\rho_k,\ k = 1,2,3$ be normalization factors:

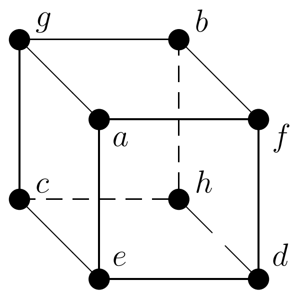
$$\rho_k = \left(\frac{\sin a_k}{2\cos\beta_0/2\cos\beta_1/2\cos\beta_2/2\cos\beta_3/2}\right)^{\frac{N-1}{N}}. \tag{3.3}$$

With these notations the vertex weight function is

$$R_{i_1,i_2,i_3}^{j_1,j_2,j_3} = \delta_{j_2+j_3,i_2+i_3}\omega^{j_3(j_1-i_1)}\rho_3\frac{w(p_1|i_1-i_2)w(p_2|j_1-j_2)}{w(p_3|i_1-j_2)w(p_4|j_1-i_2)}. \tag{3.4}$$

Such weight functions obey the Tetrahedron Equation (see section 6) and hence define an exactly solvable lattice model.

Now we will show that such a model is equivalent in the thermodynamic limit to the BBM [7]. In the BBM to each cube of the lattice we assign a weight function depending on eight corner spins (see Fig. 1).



$$= \quad W(a|e,f,g|b,c,d|h)$$

Fig. 1

We choose the weight function in the following form

$$W(a|e,f,g|b,c,d|h|a_1^B,a_2^B,a_3^B) = \\ \rho_3 \sum_\sigma \frac{w(Op_4^B|d-h-\sigma)w(Op_3^B|a-g-\sigma)}{w(Op_2^B|e-c-\sigma)w(Op_1^B|f-b-\sigma)}\omega^{\sigma(h+g-c-b)}, \tag{3.5}$$

where $p_i^B = p_i(a_1^B,a_2^B,a_3^B)$. Here a_i^B are the sides of the spherical triangle in notations of Ref. [7, 8]. In fact, expression (3.5) coincides with the weight function from Ref. [8] up to some gauge and normalization multipliers.

Consider now a chain of n weights (3.5) in the direction $a-g$ with the cyclic boundary conditions. This chain defines a weight function for some two – dimensional model, closely connected with the homogenious generalized Chiral Potts model [7, 12]. The weight function of this two – dimensional model looks like

$$\mathcal{W}(A,B,C,D|a_1^B,a_2^B,a_3^B) = \prod_\alpha W(a_\alpha|c_\alpha,b_\alpha,a_{\alpha+1}|b_{\alpha+1},c_{\alpha+1},d_\alpha|d_{\alpha+1}) = \\ \sum_{\{\sigma_\alpha\}}\prod_\alpha \rho_3 \frac{w(Op_4^B|\hat{d}_\alpha-\sigma_\alpha)w(Op_3^B|\hat{a}_\alpha-\sigma_\alpha)}{w(Op_2^B|\hat{c}_\alpha-\sigma_\alpha)w(Op_1^B|\hat{b}_\alpha-\sigma_\alpha)}\omega^{\sigma_\alpha(d_{\alpha+1}+a_{\alpha+1}-c_{\alpha+1}-b_{\alpha+1})}, \tag{3.6}$$

where the capital spin $A=\{a_\alpha\}$ and $\hat{a}_\alpha \equiv a_\alpha - a_{\alpha+1}$, etc (see Fig. 2).

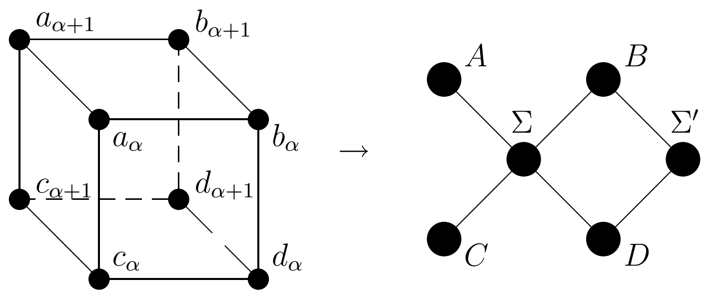


Fig. 2

Move now our frame of view to the right at the one half step of the two – dimensional lattice (see the right part of the Fig. 2). The points A and C disappear from our frame, but there will appear the right neighbour Σ' of our previous spin of the summation Σ. Then we get the four – spin weight $\mathcal{S}$:

$$S(\Sigma, B, \Sigma', D|a_1^B, a_2^B, a_3^B) =$$
$$= \prod_\alpha \rho_3 \frac{w(Op_4^B|\hat{d}_\alpha - \sigma_\alpha) w(Op_3^B|\hat{b}_\alpha - \sigma'_\alpha)}{w(Op_2^B|\hat{d}_\alpha - \sigma'_\alpha) w(Op_1^B|\hat{b}_\alpha - \sigma_\alpha)} \omega^{(\sigma_\alpha - \sigma'_\alpha)(d_{\alpha+1} - b_{\alpha+1})} \equiv$$
$$\equiv \prod_\alpha R^{-\sigma_\alpha, -\hat{d}_\alpha, b_{\alpha+1} - d_{\alpha+1}}_{-\sigma'_\alpha, -\hat{b}_\alpha, \quad b_\alpha - d_\alpha}(a_2^B, \pi - a_1^B, \pi - a_3^B), \tag{3.7}$$

where the vertex weight R is defined by (3.4). The last expression in (3.7) differs from the two – dimensional projection of the vertex lattice by a slight modification of boundary conditions. So the BBM with the weight functions (3.5), depending on a_i^B, is equivalent to the vertex model with the weight functions (3.4), depending on the a_i such as

$$a_1 = a_2^B,\ a_2 = \pi - a_1^B,\ a_3 = \pi - a_3^B \tag{3.8}$$

in the thermodynamic limit.

4. Symmetry properties.

The weight function of the BBM (3.5) is symmetric with respect to the cube symmetry group up to some multiplicative gauge transformations. In the case of the vertex weight function (3.4) corresponding gauge transformations are

the Fourier ones. To simplify all formulas we will use convenient operator notations. We will consider our weight (3.4) as an operator acting in the tensor product of three linear N – dimensional spaces so that

$$R_{i_1,i_2,i_3}^{j_1,j_2,j_3} =< i_1, i_2, i_3|R|j_1, j_2, j_3 > . \tag{4.1}$$

Define operators of the Fourier transformation and of the spin inversion:

$$< i|F|j >= \frac{1}{\sqrt{N}}\omega^{ij}, \; < i|J|j >= \delta_{i,-j}. \tag{4.2}$$

Then the inversion relation for the vertex weight is

$$\begin{aligned} R(a_1, a_2, a_3)\, J \otimes J \otimes J\, R(-a_1, -a_2, -a_3)\, J \otimes J \otimes J \;= \\ = 1 \otimes 1 \otimes 1\, \Phi(a_1, a_2, a_3), \end{aligned} \tag{4.3}$$

where

$$\Phi(a_1, a_2, a_3) = \left(\frac{\sin\beta_0/2}{\cos\beta_1/2\cos\beta_2/2\cos\beta_3/2} \right)^{2\frac{N-1}{N}} . \tag{4.4}$$

In fact, expression (4.4) coincides explicitly with the inversion factor for BBM [10].

To write down the symmetry properties of the weight (3.4) we need to define the permutation operators P_{ij}

$$P_{12}|i_1, i_2, i_3 >= |i_2, i_1, i_3 > \tag{4.5}$$

and similarly for $P_{13},\ P_{23}$.

Let t_1, t_2, t_3 be transpositions in the corresponding vector spaces and the total transposition sign t:

$$R^t(a_1, a_2, a_3) = R^{t_1t_2t_3}(a_1, a_2, a_3). \tag{4.6}$$

Then the crossing relation is

$$1 \otimes 1 \otimes J\; R^{t_1t_2}(a_1, a_2, a_3)\; 1 \otimes 1 \otimes J = R(\pi - a_1, \pi - a_2, a_3) \tag{4.7}$$

and all five space permutations are given by

$$\begin{gathered} F^{-1} \otimes F^{-1} \otimes F^{-1}R(a_1, a_2, a_3)F \otimes F \otimes F = P_{13}R^t(a_3, a_2, a_1)P_{13}, \\ J \otimes J \otimes FR(a_1, a_2, a_3)J \otimes J \otimes F^{-1} = P_{12}R^t(a_2, a_1, a_3)P_{12}. \\ F \otimes 1 \otimes 1R(a_1, a_2, a_3)F^{-1} \otimes 1 \otimes 1 = P_{23}R^t(a_1, a_3, a_2)P_{23}, \\ F^{-1} \otimes F^{-1} \otimes JR(a_1, a_2, a_3)F \otimes F \otimes J = P_{23}P_{12}R(a_3, a_1, a_2)P_{12}P_{23}, \\ J \otimes F \otimes FR(a_1, a_2, a_3)J \otimes F^{-1} \otimes F^{-1} = P_{12}P_{23}R(a_2, a_3, a_1)P_{23}P_{12}. \end{gathered} \tag{4.8}$$

Combining Fourier transformations with diagonal gauge transformations, one can obtain another forms of the R – matrix. In general these combined transformations are the gauge transformations of the lattice, but not the gauge transformations of the Tetrahedron Equation.

Note that there exists the conservation law $i_2+i_3=j_2+j_3$ for the weight (3.4). Below we write out some combined transformations which lead to another forms of the spin conservations laws:

$$N^{-1}\sum_{\alpha_3,\beta_3}\omega^{\alpha_3 j_3\,-\,\beta_3 i_3}\left(\frac{\Phi(\alpha_3)}{\Phi(\beta_3)}\right)^{\epsilon} R^{j_1,j_2,\alpha_3}_{i_1,i_2,\beta_3}(a_1,a_2,a_3)=$$
$$=\delta(j_3+j_1-i_3-i_1-\epsilon(j_2-i_2))\Phi(j_2-i_2)^{-\epsilon}\omega^{-i_3(j_2\,-\,i_2)}\times$$
$$\times\rho_3\,\frac{w(p_1(a_1,a_2,a_3)|i_1-i_2)w(p_2(a_1,a_2,a_3)|j_1-j_2)}{w(p_3(a_1,a_2,a_3)|i_1-j_2)w(p_4(a_1,a_2,a_3)|j_1-i_2)}, \tag{4.9}$$

$$N^{-2}\sum_{\alpha_m,\beta_m}\omega^{\alpha_1 j_1\,-\,\beta_1 i_1\,+\,\alpha_2 j_2\,-\,\beta_2 i_2}\left(\frac{\Phi(\alpha_2)}{\Phi(\beta_2)}\right)^{\epsilon} R^{\alpha_1,\alpha_2,j_3}_{\beta_1,\beta_2,i_3}(a_1,a_2,a_3)=$$
$$=\delta(j_2+j_1-i_2-i_1-\epsilon(j_3-i_3))\Phi(j_3-i_3)^{\epsilon}\omega^{j_2(i_3\,-\,j_3)}\times$$
$$\times\rho_2\,\frac{w(p_1(a_1,a_3,a_2)|-j_1-j_3)w(p_2(a_1,a_3,a_2)|-i_1-i_3)}{w(p_3(a_1,a_3,a_2)|-j_1-i_3)w(p_4(a_1,a_3,a_2)|-i_1-j_3)}, \tag{4.10}$$

$$N^{-3}\sum_{\alpha_m,\beta_m}(\prod_m\omega^{\alpha_m j_m\,-\,\beta_m i_m})\left(\frac{\Phi(\alpha_1)}{\Phi(\beta_1)}\right)^{\epsilon} R^{\alpha_1,\alpha_2,\alpha_3}_{\beta_1,\beta_2,\beta_3}(a_1,a_2,a_3)=$$
$$=\delta(j_1+j_2-i_1-i_2-\epsilon(j_3-i_3))\Phi(j_3-i_3)^{-\epsilon}\omega^{i_1(i_3\,-\,j_3)}\times$$
$$\times\rho_1\,\frac{w(p_1(a_3,a_2,a_1)|j_3-j_2)w(p_2(a_3,a_2,a_1)|i_3-i_2)}{w(p_3(a_3,a_2,a_1)|j_3-i_2)w(p_4(a_3,a_2,a_1)|i_3-j_2)}, \tag{4.11}$$

where $\delta(a)=\delta_{a,0}$, $\Phi(a)$ and ρ_k are defined by (2.12, 3.3) and ϵ is an arbitrary integer. Another choices of the diagonal Φ factors lead us to complicated nonmultiplicative expressions for the weights. The exception is the case $N=2$.

5. The case $N=2$

In this section we consider the case $N=2$ in a special gauge in which our R matrix (3.4) can be reduced to the vertex solutions of Korepanov and

Hietarinta in static and planar limits correspondingly.

For the case $N = 2$ the list of suitable Fourier transformations enlarges. Namely,

$$
\begin{aligned}
N^{-3}\xi \sum_{\alpha_m,\beta_m} \left(\prod_{m=1}^{3} \omega^{\alpha_m j_m - \beta_m i_m} \frac{\Phi(\beta_m)}{\Phi(\alpha_m)} \right) R^{\alpha_1,\alpha_2,\alpha_3}_{\beta_1,\beta_2,\beta_3}(a_1, a_2, a_3) = \\
= \delta(i_1 + j_1 + i_3 + j_3 - i_2 - j_2)\omega^{(i_3 - j_3)(i_2 - j_3)} \times \\
\times \frac{w(r_1| - i_1 + j_2 - j_3)w(r_3|i_1 - i_2 + j_3)}{w(Or_0|i_1 - i_2 + i_3)w(Or_2| - i_1 + j_2 - i_3)} \equiv \overline{R}^{j_1,j_2,j_3}_{i_1,i_2,i_3},
\end{aligned}
\tag{5.1}
$$

where $\omega = -1$,

$$
\xi = \left(4\sqrt{\cot\beta_0/2 \ldots \cot\beta_3/2} \right)^{\frac{N-1}{N}}, \tag{5.2}
$$

and four points r_i are given by

$$
\begin{aligned}
x(r_i) = \exp(-i\beta_i/N),\ y(r_i) = \omega^{1/4}\sqrt[N]{2\sin\beta_i}, \\
z(r_i) = \exp(i\beta_i/N),\ i = 0, 1, 2, 3.
\end{aligned}
\tag{5.3}
$$

Due to the total symmetry, this transformation is the gauge transformation of the Tetrahedron Equation, so this weight $\overline{R}$ obeys the Tetrahedron Equation (see the section 6).

When $N = 2$ the function w is very simple:

$$
\frac{w(r_i|1)}{w(r_i|0)} = \exp(i\frac{\pi}{4})\sqrt{\tan\frac{\beta_i}{2}}. \tag{5.4}
$$

Define

$$
\sqrt{\tan\frac{\beta_i}{2}} = t_i,\ i = 0, 1, 2, 3. \tag{5.5}
$$

Then we can represent the weights (5.1) by the following compact table:

$$
\begin{array}{|ccccccccc|}
\hline
\overline{R}_{0,0,0}^{0,0,0} & = & \overline{R}_{0,1,1}^{0,1,1} & = & \overline{R}_{1,0,1}^{1,0,1} & = & \overline{R}_{1,1,0}^{1,1,0} & = & 1 \\
\overline{R}_{1,1,1}^{1,1,1} & = & \overline{R}_{1,0,0}^{1,0,0} & = & \overline{R}_{0,1,0}^{0,1,0} & = & \overline{R}_{0,0,1}^{0,0,1} & = & t_0t_1t_2t_3 \\
\hline
\overline{R}_{0,0,1}^{0,1,0} & = & \overline{R}_{0,1,0}^{0,0,1} & = & -\overline{R}_{1,1,1}^{1,0,0} & = & -\overline{R}_{1,0,0}^{1,1,1} & = & t_0t_1 \\
\overline{R}_{1,1,0}^{1,0,1} & = & \overline{R}_{1,0,1}^{1,1,0} & = & -\overline{R}_{0,0,0}^{0,1,1} & = & -\overline{R}_{0,1,1}^{0,0,0} & = & t_2t_3 \\
\hline
\overline{R}_{0,1,0}^{1,1,1} & = & \overline{R}_{0,0,1}^{1,0,0} & = & -\overline{R}_{1,0,0}^{0,0,1} & = & -\overline{R}_{1,1,1}^{0,1,0} & = & it_0t_2 \\
\overline{R}_{1,0,1}^{0,0,0} & = & \overline{R}_{1,1,0}^{0,1,1} & = & -\overline{R}_{0,1,1}^{1,1,0} & = & -\overline{R}_{0,0,0}^{1,0,1} & = & -it_1t_3 \\
\hline
\overline{R}_{1,1,1}^{0,0,1} & = & \overline{R}_{0,0,1}^{1,1,1} & = & \overline{R}_{0,1,0}^{1,0,0} & = & \overline{R}_{1,0,0}^{0,1,0} & = & t_0t_3 \\
\overline{R}_{0,0,0}^{1,1,0} & = & \overline{R}_{1,1,0}^{0,0,0} & = & \overline{R}_{1,0,1}^{0,1,1} & = & \overline{R}_{0,1,1}^{1,0,1} & = & t_1t_2 \\
\hline
\end{array}
$$

The weight, defined by this table, differs from (5.1) by some normalization factor. Using the following property of a spherical triangle:

$$\tan\frac{\beta_i}{2}\tan\frac{\beta_j}{2} = \tan\frac{\alpha_k}{2}\tan\frac{\alpha_l}{2},\ \{i,j,k,l\} = \{0,1,2,3\}, \tag{5.6}$$

where α_i are the angle excesses of the spherical triangle, we can easily obtain the static limit of the $\overline{R}$ (the case when $\alpha_0 = 0$). This static limit appears to be the solution of the Tetrahedron Equation proposed by Korepanov [4]. Moreover, in the planar limit when $\beta_2 = 0$ the vertex weight (5.1) coincides with the $N = 2$ solution by Hietarinta [5, 6].

6. The Tetrahedron Equation.

The vertex form of the TE is the following:

$$\sum_{\substack{k_1,k_2,k_3,\\k_4,k_5,k_6}} R^{k_1,k_2,k_3}_{i_1,i_2,i_3} R'^{j_1k_4k_5}_{k_1i_4i_5} R''^{j_2j_4k_6}_{k_2k_4i_6} R'''^{j_3j_5j_6}_{k_3k_5k_6} =$$
$$= \sum_{\substack{k_1,k_2,k_3,\\k_4,k_5,k_6}} R'''^{k_3,k_5,k_6}_{i_3,i_5,i_6} R''^{k_2k_4j_6}_{i_2i_4k_6} R'^{k_1j_4j_5}_{i_1k_4k_5} R^{j_1j_2j_3}_{k_1k_2k_3} \quad . \tag{6.1}$$

A complete solution of this equation is parameterized by six angles of a tetrahedron (five of them are independent):

$$\begin{aligned} R &= R(\theta_1,\theta_2,\theta_3),\\ R' &= R(\theta_1,\theta_4,\theta_5),\\ R'' &= R(\pi-\theta_2,\theta_4,\theta_6)\\ R''' &= R(\theta_3,\pi-\theta_5,\theta_6). \end{aligned} \tag{6.2}$$

The ordering of the dihedral angles is natural with respect to the numbering of the spaces and differs from that in the standard equation (2.2) in Ref. [9].

For each vertex in (6.1) let a_i be the corresponding planar angles:

$$\begin{aligned} (\theta_1,\theta_2,\theta_3) &\to (a_1,a_2,a_3),\\ (\theta_1,\theta_4,\theta_5) &\to (a'_1,a'_2,a'_3),\\ (\pi-\theta_2,\theta_4,\theta_6) &\to (a''_1,a''_2,a''_3),\\ (\theta_3,\pi-\theta_5,\theta_6) &\to (a'''_1,a'''_2,a'''_3), \end{aligned} \tag{6.3}$$

Then planar angles of four weights are constrained as follows

$$\begin{aligned} a''_3 &= a'_3 - a_3, \quad a'''_1 = a''_1 - a'_1,\\ a'''_2 &= a''_2 - a_1, \quad a'''_3 = a'_2 - a_2. \end{aligned} \tag{6.4}$$

Four vertex weights (3.4) with the angles defined by (6.3) and satisfying to (6.4) obey Tetrahedron Equation (6.1). In this section we give a sketch of the proof of this statement.

Let us substitute (3.4) in (6.1). Due to the spin conservation laws, six summations in the both sides reduce to three ones. It is useful to choose the indices k_1, k_2, k_4 as independent spins of the summations. The summands in

the left and right hand sides are the products of the phase factors $\omega^{\cdots}$ and w functions. First let us collect in both sides all factors depending on the spin k_1. The sums over k_1 have the form ${}_2\Psi_2$ (see Appendix). As the first step let us apply to these ${}_2\Psi_2$ $(\tau\rho)^2$ transformations (see formula (A.14) from Appendix). As a result there appear extra w functions, depending on k_2-k_4, and we demand a cancellation of these extra factors with similar w functions in the left and right hand sides. It gives us some algebraic constraints on parameters of weights.

Further summations over k_2 and k_4 become independent. Moreover, there are no phase factors, depending on k_2 and k_4, and we can sum over k_2 and k_4 using the "Star – Square" relation (see formula (A.19) from Appendix). Finally sums over k_1 are both of the type ${}_4\Psi_4$ and have a similar spin structure. Imposing necessary constraints among parameters of weights we come to the equality of the left and right hand sides of the TE.

We will not write out here algebraic constraints coming from the cancellation of all w functions, depending on k_2-k_4, the "Star – Square" applicability conditions (A.20) and the coincidence of the final expressions. All calculations are direct but rather tedious.

As a result we obtain that all restrictions on parameters of weight functions are satisfied automatically if we take into account parameterization (3.1) and constraints between angles (6.3-6.4).

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Appendix

In this Appendix we collect useful formulae in the theory of ω – hypergeometric seria with ω being a N-th root of unity. In fact, these formulae (or their particular cases) appeared in many papers devoted to the Chiral Potts

Model [14–16] and to the TE. Let us define ${}_r\Psi_r$ series as

$$
{}_r\Psi_r\left(\begin{matrix}(p_1,m_1)\dots(p_r,m_r)\\(p'_1,m'_1)\dots(p'_r,m'_r)\end{matrix}\middle|\,n\right)=\sum_{\sigma\in Z_N}\frac{w(p_1|m_1+\sigma)\dots w(p_r|m_r+\sigma)}{w(p'_1|m'_1+\sigma)\dots w(p'_r|m'_r+\sigma)}\frac{\omega^{n\sigma}}{\sqrt{N}}. \tag{A.1}
$$

Discuss now a role of the normalization. Spin independent factors in all identities in this Appendix are given for the case when all arguments of the w functions in the left and right hand sides belong to the region $\overline{\Gamma}_0$ (see (2.8)). If we abandon restriction (2.8) then the phases of w's can be chosen in such a way that the corresponding formulae will still remain correct.

To simplify all notations we will omit arguments in the components of points p_i and imply that

$$
p_i=(x_i,y_i,z_i) \tag{A.2}
$$

for every i. In last formulas of this Appendix we will use also points q_i. In this case we will explicitly point out by upper subscript the corresponding point

$$
q_i=(x_i^q,y_i^q,z_i^q). \tag{A.3}
$$

There will appear many new points on the Fermat curve in right hand sides of formulae. In these cases we have to introduce new letters for y components. They have to be defined by (2.4) (and belong to region (2.9) in the accordance with our agreement).

All formulae in this Appendix are formulae of the summation (they exist for $r=1,2,3$) and symmetry transformation formulae (they exist for $r=1,2,3,4$).

We begin with a cyclic analog of Ramanujan summation formula for $r=1$. In fact, this is nothing else as restricted Star-Triangle relation of Bazhanov-Baxter model (see Ref. [13])

$$
{}_1\Psi_1\left(\begin{matrix}(p_1,m_1)\\(p_2,m_2)\end{matrix}\middle|\,n\right)= \tag{A.4}
$$

$$
=\Phi_0\left(\frac{\xi}{y_1y_2}\right)^{\frac{N-1}{2}}\frac{w(z_1y_2,\xi,y_1z_2|-n)w(x_1z_2,\xi,\omega z_1x_2|m_1-m_2)}{\omega^{nm_2}\,w(x_1y_2,\xi,\omega y_1x_2|m_1-m_2-n)}=
$$

$$
=\left(\frac{\omega^{-1/2}\xi}{y_1y_2}\right)^{\frac{N-1}{2}}\frac{\Phi_0^{-1}\,\omega^{-nm_1}\,w(y_1x_2,\omega^{-1/2}\xi,x_1y_2|m_2-m_1+n)}{w(y_1z_2/\omega,\omega^{-1/2}\xi,z_1y_2|n)w(z_1x_2,\omega^{-1/2}\xi,x_1z_2|m_2-m_1)},
$$

where

$$\Phi_0 = \exp\Big\{i\pi\frac{(N-1)(N-2)}{12N}\Big\}. \tag{A.5}$$

For the proof of this formula see, for example, Ref. [11, 13].

Here we give also two forms of inversion relations for w functions which have been used for the proof of inversion relation (4.3):

$$\sum_{\sigma}\frac{w(\omega x,\omega y,z|m_1+\sigma)}{w(x,y,z|m_2+\sigma)} = N\delta_{m_1,\,m_2}\left(\frac{\omega^{1/2}x}{y}\right)^{N-1}. \tag{A.6}$$

$$\sum_{\sigma}\frac{w(x,y,z|m_1+\sigma)}{w(x,y,\omega z|m_2+\sigma)} = N\delta_{m_1,\,m_2}\left(\frac{\omega^{1/2}x}{y}\right)^{N-1}. \tag{A.7}$$

Note that points $(\omega x,\omega y,z)$ and $(x,y,\omega z)$ do not belong to the region $\overline{\Gamma}_0$ and we define for these cases

$$\frac{w(\omega x,\omega y,z|0)}{w(x,y,z|0)} = -\omega^{1/2}\frac{y}{z-\omega x} \tag{A.8}$$

and

$$\frac{w(x,y,\omega z|0)}{w(x,y,z|0)} = -\omega^{1/2}\frac{z-x}{y}, \tag{A.9}$$

where $(x,y,z)\in\overline{\Gamma}_0$.

To obtain symmetry formulae for higher r, we use the following simple fact. Let g_1 and g_2 be arbitrary functions on Z_N. If

$$\tilde{g}_i(k) = \sum_{\sigma\in Z_N} g_i(\sigma)\frac{\omega^{k\sigma}}{\sqrt{N}}, \tag{A.10}$$

then

$$\sum_{\sigma\in Z_N} g_1(\sigma)g_2(\sigma) = \sum_{\sigma\in Z_N}\tilde{g}_1(\sigma)\tilde{g}_2(-\sigma). \tag{A.11}$$

Using this, it is easily to obtain the following symmetry transformation for ${}_2\Psi_2$:

$$
{}_2\Psi_2\left(\begin{matrix}(p_1,m_1)(p_3,m_3)\\(p_2,m_2)(p_4,m_4)\end{matrix}\middle|\, n\right) = {}_2\Psi_2\left(\begin{matrix}(q_1,0)\;(q_3,m_4-m_3+n)\\(q_2,n)\quad\;(q_4,m_1-m_2)\end{matrix}\middle|\, m_2-m_3\right)
$$

$$
\times\left(\frac{\xi_{12}\xi_{43}}{y_1y_2y_3y_4}\right)^{\frac{N-1}{2}}\omega^{-nm_3}\frac{w(x_1z_2,\xi_{12},\omega z_1x_2|m_1-m_2)}{w(z_3x_4,\xi_{43},x_3z_4|m_4-m_3)}, \tag{A.12}
$$

where

$$
\begin{aligned}
q_1 &= (z_1y_2,\xi_{12},y_1z_2), \qquad & q_3 &= (y_3x_4,\xi_{43},x_3y_4),\\
q_2 &= (y_3z_4/\omega,\xi_{43},z_3y_4), \qquad & q_4 &= (x_1y_2,\xi_{12},\omega y_1x_2).
\end{aligned} \tag{A.13}
$$

This relation has appeared originally as the $(\tau\rho)$ transformation in Ref. [11] for the BB weight function. Note that $(\tau\rho)^6=1$. In this paper we have used $(\tau\rho)^2$ transformation:

$$
{}_2\Psi_2\left(\begin{matrix}(p_1,m_1)(p_3,m_3)\\(p_2,m_2)(p_4,m_4)\end{matrix}\middle|\, n\right) =
$$

$$
= {}_2\Psi_2\left(\begin{matrix}(s_1,0) & (s_3,m_1-m_4-n)\\(s_2,m_2-m_3) & (s_4,-n)\end{matrix}\middle|\, m_3-m_4\right)\times
$$

$$
\times\,\omega^{-nm_2-(m_2-m_3)(m_4-m_3)}\left(\frac{\Lambda\Lambda'}{\xi_{12}\xi_{43}y_1y_2y_3y_4}\right)^{\frac{N-1}{2}}\times
$$

$$
\times\frac{w(x_1z_2,\xi_{12},\omega z_1x_2|m_1-m_2)}{w(z_3x_4,\xi_{43},x_3z_4|m_4-m_3)}\frac{w(z_1z_3y_2y_4,\Lambda,z_2z_4y_1y_3|-n)}{w(x_1x_3y_2y_4,\Lambda',\omega x_2x_4y_1y_3|-\overline{n})}, \tag{A.14}
$$

where

$$
\overline{n}=n-m_1-m_3+m_2+m_4, \tag{A.15}
$$

and

$$
\begin{aligned}
s_1 &= (y_1z_2\xi_{43},\Lambda,z_3y_4\xi_{12}), \qquad & s_3 &= (x_1y_2\xi_{43},\Lambda',y_3x_4\xi_{12}),\\
s_2 &= (y_1x_2\xi_{43},\Lambda',x_3y_4\xi_{12}), \qquad & s_4 &= (z_1y_2\xi_{43},\Lambda,y_3z_4\xi_{12}).
\end{aligned} \tag{A.16}
$$

The list of the symmetry formulae for ${}_2\Psi_2$ we finish by the $T=(\tau\rho)^3$:

$$
{}_2\Psi_2\left(\begin{matrix}(p_1,m_1)(p_3,m_3)\\(p_2,m_2)(p_4,m_4)\end{matrix}\middle|\, n\right) = {}_2\Psi_2\left(\begin{matrix}(\overline{p}_1,-m_3)(\overline{p}_3,-m_1)\\(\overline{p}_2,-m_4)(\overline{p}_4,-m_2)\end{matrix}\middle|\, \overline{n}\right)\times
$$

$$\times\, \omega^{-nm_2 - m_4\overline{n}} \left(\frac{\xi_{12}\xi_{43}\xi_{32}\xi_{41}}{\Lambda\Lambda'}\right)^{\frac{N-1}{2}} \frac{w(x_1z_2, \xi_{12}, \omega z_1x_2|m_1 - m_2)}{w(z_3x_4, \xi_{43}, x_3z_4|m_4 - m_3)} \times$$
$$\times \frac{w(z_1z_3y_2y_4, \Lambda, z_2z_4y_1y_3| - n)}{w(x_1x_3y_2y_4, \Lambda', \omega x_2x_4y_1y_3| - \overline{n})} \frac{w(x_3z_2, \xi_{32}, \omega z_3x_2|m_3 - m_2)}{w(z_1x_4, \xi_{41}, x_1z_4|m_4 - m_1)}, \quad \text{(A.17)}$$

where

$$\overline{p}_1 = (z_3\Lambda', \xi_{32}\xi_{43}y_1, x_3\Lambda), \qquad \overline{p}_3 = (z_1\Lambda', \xi_{41}\xi_{12}y_3, x_1\Lambda),$$
$$\overline{p}_2 = (z_4\Lambda', \omega\xi_{41}\xi_{43}y_2, \omega x_4\Lambda), \qquad \overline{p}_4 = (z_2\Lambda', \xi_{32}\xi_{12}y_4, \omega x_2\Lambda). \quad \text{(A.18)}$$

Note that in the case when $\frac{x_1x_3}{x_2x_4} = \omega\frac{z_1z_3}{z_2z_4}$, (A.17) becomes the Star – Star relation for the BBM, and $p_i = \overline{p}_i$.

To obtain a summation formula for ${}_2\Psi_2$, consider (A.12) and set $n = 0$ and $q_1 = q_2$. Then applying (A.4) to the right hand side of (A.12), we obtain "Star – Square" relation:

$$ {}_2\Psi_2\left(\begin{matrix}(p_1, m_1)(p_3, m_3)\\(p_2, m_2)(p_4, m_4)\end{matrix}\middle| 0\right) = \left(\frac{\omega^{1/2}\Lambda'}{y_1y_2y_3y_4}\right)^{\frac{N-1}{2}} \omega^{-(m_2 - m_3)(m_1 - m_2)} \times$$
$$\times\, \Phi_0\, w(x_2x_4y_1y_3, \omega^{-1/2}\Lambda', x_1x_3y_2y_4|m_2 + m_4 - m_1 - m_3)\times$$
$$\times \frac{w(x_1z_2, \xi_{12}, \omega z_1x_2|m_1 - m_2)}{w(z_3x_4, \xi_{43}, x_3z_4|m_4 - m_3)} \frac{w(x_3z_2, \xi_{32}, \omega z_3x_2|m_3 - m_2)}{w(z_1x_4, \xi_{41}, x_1z_4|m_4 - m_1)}, \quad \text{(A.19)}$$

where the parameters in the left hand side have to obey a special restriction:

$$\frac{y_1y_3}{y_2y_4} = \omega\frac{z_1z_3}{z_2z_4}, \quad \text{(A.20)}$$

and the phases in the right hand side are constrained by

$$\frac{\xi_{12}}{\xi_{43}} = \frac{y_1z_2}{y_4z_3}, \quad \frac{\xi_{32}}{\xi_{41}} = \frac{y_3z_2}{y_4z_1}, \quad \Lambda' = \omega^{-1/2}\xi_{12}\xi_{32}\frac{y_4}{z_2}. \quad \text{(A.21)}$$

Further we will try to avoid such long notations as in (A.17) and (A.19). Extra w multipliers in all consequent formulae will have the same structure as in the right hand side of (A.17) and so we will use only ξ_{ij} to denote a whole argument dependence of w.

Consider now $r = 3$. A summation formula can be obtained summing (A.17) over n with the help of the restricted Star-Triangle relation (A.4).

The result reads

$$
{}_3\Psi_3\left(\left.\begin{matrix}(p_1,m_1)(p_3,m_3)(q_1,m_2+m_4-\lambda)\\(p_2,m_2)(p_4,m_4)(q_2,m_1+m_3-\lambda)\end{matrix}\right|0\right)=
$$

$$
=\Phi_0^{-1}\left(\frac{\xi_{12}\xi_{43}\xi_{32}\xi_{41}}{y_1y_2y_3y_4\Xi}\right)^{\frac{N-1}{2}}\times
$$

$$
\times\frac{\omega^{(m_4-\lambda)(m_1+m_3-m_2-m_4)}}{w(x_1x_3z_2z_4,\Xi,\omega^2x_2x_4z_1z_3|m_1+m_3-m_2-m_4)}\times
$$

$$
\times\frac{w(\xi_{12}|m_1-m_2)w(\xi_{32}|m_3-m_2)}{w(\xi_{43}|m_4-m_3)w(\xi_{41}|m_4-m_1)}\frac{w(\overline{p}_1|\lambda-m_3)w(\overline{p}_3|\lambda-m_1)}{w(\overline{p}_2|\lambda-m_4)w(\overline{p}_4|\lambda-m_2)},\quad (A.22)
$$

where $w(\xi_{ij})$ and $w(\overline{p}_i)$ are the same as in (A.17) and

$$
\begin{aligned}
q_1&=(\omega x_2x_4\Lambda,y_2y_4\Xi,z_2z_4\Lambda'),\\
q_2&=(x_1x_3\Lambda,y_1y_3\Xi,\omega z_1z_3\Lambda').
\end{aligned}\quad (A.23)
$$

Note, that the formula (A.22) is symmetric with respect to any permutation of p_1,p_3,q_1 and p_2,p_4,q_2. The Star – Triangle relation [15, 16] for the Chiral Potts Model is a special case of (A.22).

To obtain symmetry relations for $r=3$ and $r=4$, we have to use (A.11), apply (A.4) or T transformation correspondingly, cancel extra w factors (this gives some constraints) and then, using (A.11) again, obtain the corresponding ${}_r\Psi_r$ in the right hand side. The formula for $r=3$ reads

$$
{}_3\Psi_3\left(\left.\begin{matrix}(p_1,m_1)(p_3,m_3)(q_1,l_1)\\(p_2,m_2)(p_4,m_4)(q_2,l_2)\end{matrix}\right|0\right)=
$$

$$
=\left(\frac{\xi_{12}\xi_{43}\xi_{32}\xi_{41}\xi}{\Lambda'^2\Lambda y_1^q y_2^q}\right)^{\frac{N-1}{2}}\omega^{(l_1-m_2)(m_1+m_3-m_2-m_4)}\times
$$

$$
\times\frac{w(\xi_{12}|m_1-m_2)w(\xi_{32}|m_3-m_2)}{w(\xi_{43}|m_4-m_3)w(\xi_{41}|m_4-m_1)}\frac{w(\xi|l_2+m_2+m_4-l_1-m_1-m_3)}{w(\lambda|l_2-l_1)}\times
$$

$$
\times{}_3\Psi_3\left(\left.\begin{matrix}(\overline{p}_1,-m_3)(\overline{p}_3,-m_1)(\overline{q}_1,l_1-m_2-m_4)\\(\overline{p}_2,-m_4)(\overline{p}_4,-m_2)(\overline{q}_2,l_2-m_1-m_3)\end{matrix}\right|0\right),\quad (A.24)
$$

where the connections between arguments in the left hand side are

$$
\frac{y_1^py_3^py_1^q}{y_2^py_4^py_2^q}=\omega\frac{z_1^pz_3^pz_1^q}{z_2^pz_4^pz_2^q},\qquad \frac{\Lambda}{\lambda}=\frac{z_2^pz_4^py_1^py_3^p}{z_1^qy_2^q}\quad (A.25)
$$

and new arguments in the right hand side of (A.24) are

$$\begin{aligned}(\xi) &= (\omega x_2^p x_4^p x_2^q y_1^p y_3^p y_1^q, \xi, \omega x_1^p x_3^p x_1^q y_2^p y_4^p y_2^q),\\ (\lambda) &= (z_1^q x_2^q, \lambda, x_1^q z_2^q),\\ \overline{q}_1 &= (x_1^q y_2^q \Lambda', \xi, \omega\lambda x_2^p x_4^p y_1^p y_3^p),\\ \overline{q}_2 &= (y_1^q x_2^q \Lambda', \xi, \omega\lambda x_1^p x_3^p y_2^p y_4^p).\end{aligned} \tag{A.26}$$

Note that this formula is a symmetry transformation for something. Denote (A.24) as ρ_3. Let τ_3 be a permutation transformation, reordering the columns in ${}_3\Psi_3$ as $\tau_3(1,2,3) = (2,3,1)$. Then $(\tau_3\rho_3)^6 = 1$.

The last formula is a symmetry transformation for ${}_4\Psi_4$. A derivation of it is described before formula (A.24). Let the structure of a set $q_i, \overline{q}_i, \chi_{ij}, \Delta, \Delta'$ is defined identically to that of $p_i, \overline{p}_i, \xi_{ij}, \Lambda, \Lambda'$. Then

$$\begin{aligned}&{}_4\Psi_4\left(\begin{matrix}(p_1,m_1)(p_3,m_3)(q_1,l_1)(q_3,l_3)\\(p_2,m_2)(p_4,m_4)(q_2,l_2)(q_4,l_4)\end{matrix}\middle|\,0\right) = \\ &= \left(\frac{\xi_{12}\xi_{43}\xi_{32}\xi_{41}\chi_{12}\chi_{43}\chi_{32}\chi_{41}}{\Lambda'\Lambda\Delta'\Delta}\right)^{\frac{N-1}{2}} \frac{\omega^{(l_2-m_2)(m_1+m_3-m_2-m_4)}}{\Phi(m_1+m_3-m_2-m_4)}\times \\ &\times\frac{w(\xi_{12}|m_1-m_2)w(\xi_{32}|m_3-m_2)}{w(\xi_{43}|m_4-m_3)w(\xi_{41}|m_4-m_1)}\frac{w(\chi_{12}|l_1-l_2)w(\chi_{32}|l_3-l_2)}{w(\chi_{43}|l_4-l_3)w(\chi_{41}|l_4-l_1)}\times \\ &\times{}_4\Psi_4\left(\begin{matrix}(\overline{p}_1,-m_3)(\overline{p}_3,-m_1)(\overline{q}_1,l_1-m_2-m_4)(\overline{q}_3,l_3-m_2-m_4)\\(\overline{p}_2,-m_4)(\overline{p}_4,-m_2)(\overline{q}_2,l_2-m_1-m_3)(\overline{q}_4,l_4-m_1-m_3)\end{matrix}\middle|\,0\right),\end{aligned} \tag{A.27}$$

where the constraints are

$$\frac{y_1^p y_3^p y_1^q y_3^q}{y_2^p y_4^p y_2^q y_4^q} = \omega\frac{z_1^p z_3^p z_1^q z_3^q}{z_2^p z_4^p z_2^q z_4^q} = \omega^{-1}\frac{x_1^p x_3^p x_1^q x_3^q}{x_2^p x_4^p x_2^q x_4^q} \tag{A.28}$$

and

$$\frac{\Lambda}{\Delta} = \omega^{-1/2}\frac{y_1^p y_3^p z_2^p z_4^p}{z_1^q z_3^q y_2^q y_4^q}, \quad \frac{\Lambda'}{\Delta'} = \omega^{1/2}\frac{y_1^p y_3^p x_2^p x_4^p}{x_1^q x_3^q y_2^q y_4^q}, \tag{A.29}$$

and the spins in (A.27) are not independent:

$$m_1 + m_3 + l_1 + l_3 = m_2 + m_4 + l_2 + l_4. \tag{A.30}$$

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